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DEDEKIND DIFFERENT AND TYPE SEQUENCE

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Dedicated to Silvio Greco in occasion of his 60-th birthday.

Let R be a one-dimensional, local, Noetherian domain. We assume R analitycally irreducible and residually rational. Let ω be a *canonical module* of R such that $R \subseteq \omega \subseteq \overline{R}$ and let $\theta_D := R : \omega$ be the *Dedekind different* of R.

Our purpose is to study how θ_D is involved in the type sequence of R and to compare the type sequence of R with the type sequence of θ_D (for the notion of type sequence we refer to [11], [1] and [13]). These relations yield some interesting consequences.

1. Introduction.

Let $(R, \mathfrak{m})$ be a one-dimensional, local, Noetherian domain and let $\overline{R}$ be the integral closure of R in its quotient field K. We assume that $\overline{R}$ is a DVR and a finite R-module, which means that R is analitycally irreducible. Let $t \in \overline{R}$ be a uniformizing parameter for $\overline{R}$, so that $t\overline{R}$ is the maximal ideal of $\overline{R}$. We also suppose R to be residually rational, i.e. $R/\mathfrak{m} \simeq \overline{R}/t\overline{R}$.

In our hypotheses there exists a *canonical module* of R unique up to isomorphism, namely a fractional ideal ω such that $\omega : (\omega : I) = I$ for each fractional ideal I of R. We can assume that $R \subseteq \omega \subset \overline{R}$. The *Dedekind different of* R is the ideal $\theta_D := R : \omega$.

Let $\nu : K \longrightarrow \mathbb{Z} \cup \infty$ be the usual valuation associated to $\overline{R}$. The image $\nu(R) = \{\nu(x),\ x \in R,\ x \neq 0\} \subseteq \mathbb{N}$ is a numerical semigroup of $\mathbb{N}$.

The *multiplicity* of R is the smallest non-zero element e in $\nu(R)$. The *conductor* of $\nu(R)$ is the minimal $c \in \nu(R)$ such that every $m \geq c$ is in $\nu(R)$ and $\gamma := t^c\overline{R}$ is the *conductor ideal* of R. We denote by δ the classical *singularity degree*, that is the number of gaps of the semigroup $\nu(R)$ in $\mathbb{N}$.

We briefly recall the notion of *type sequence* given for rings in [11], recently revisited in [1] and extended to modules in [13].

Let $n = c - \delta$, and call $s_0 = 0, s_1, \ldots, s_n = c$ the first $n+1$ elements of $\nu(R)$. Form the chain of ideals $R_0 \supset R_1 \supset R_2 \supset \ldots \supset R_n$, where, for each i, $R_i := \{x \in R \ : \ \nu(x) \geq s_i\}$.
Note that $R = R_0$, $R_1 = \mathfrak{m}$, $R_n = \gamma$.
Now construct the two chains:

$$R = R : R_0 \subset R : \mathfrak{m} \subset R : R_2 \subset \ldots \subset R : R_n = \overline{R}$$

$$\theta_D = \theta_D : R_0 \subset \theta_D : \mathfrak{m} \subset \theta_D : R_2 \subset \ldots \subset \theta_D : R_n = \overline{R}$$

For every $i = 1 \ldots n$, define

$$r_i = l_R(R : R_i/R : R_{i-1}) = l_R(\omega R_{i-1}/\omega R_i),$$

$$t_i = l_R(\theta_D : R_i/\theta_D : R_{i-1}) = l_R(\omega^2 R_{i-1}/\omega^2 R_i).$$

The *type sequence* of R, denoted by $t.s.(R)$, is the sequence $[r_1, \ldots, r_n]$. The *type sequence* of θ_D, denoted by $t.s.(\theta_D)$, is the sequence $[t_1, \ldots, t_n]$. Observe that r_1 is the *Cohen Macaulay type* of R which is also the minimal number of generators of ω and that t_1 is the *C.M. type* of the R-module θ_D, or the minimal number of generators of ω^2. Moreover, for every i, we have $1 \leq r_i \leq r_1$ and $1 \leq t_i \leq t_1$ (see e.g. [13], Prop. 1.6, for all details).

We show in Prop. 3.4 that, if $s_i \in \nu(\theta_D)$, then the correspondent $r_i + 1$ is 1. Hence, denoting by p the number of 1's in the type sequence of R, we get (see Theorem 3.7) the inequalities

$$\delta \leq (c-\delta)r_1 - p(r_1 - 1) \ \leq \ (c-\delta)r_1 - l_R(\theta_D/\gamma)(r_1 - 1)$$

which improve the well known formula $\delta \leq (c-\delta)r_1$ (see Remark 3.12).

A ring R is called *almost Gorenstein ring* if its type sequence is of the kind $[r_1, 1, \ldots, 1]$; in the general case we focus our attention to the last i such that $r_i > 1$, and we show its special meaning related to the blowing up of the canonical module and to the Dedekind different (Theorem 4.3). An easy corollary is the inequality $l_R(R/\theta_D) \leq i$.

We compare the two type sequences in several cases. For instance, in a ring R of CM type 2 they can be completely determined by using the Dedekind different (Prop. 4.10). Under suitable hypotheses we have that $r_i \leq t_i$, although this is not always true. We conjecture however that $r_1 \leq t_1$ always holds and we can prove this inequality in the following cases:

- R is almost Gorenstein (see Prop. 5.1);

- R has C.M. type 2, 3, $e-1$ (see Prop. 4.10, Corollary 3.9, Prop. 4.9);
- $\theta_D = \gamma$ (see Prop. 4.8);
- R satisfies the inequality $l_R(R/\theta_D)(r_1 - 2) \leq 2\delta - c$ (see Prop. 4.11).

In section 5 some results are achieved for minimal and maximal type sequences. In particular in Prop. 5.1, we prove that R is a *almost Gorenstein ring*, (that is $t.s.(R)$ is minimal), if and only if $t.s.(\theta_D)$ is also minimal. On the other side we prove in Prop. 5.4, that the $t.s.(R)$ is maximal, i.e. of the kind $[e-1, ..., e-1, e-1-a]$ for some $a < e-2$ or of the kind $[e-1, ..., e-1,\ 1]$, if and only if $t.s.(\theta_D)$ is maximal, i.e. of the kinds $[e, e, ..., e, e-a]$, $[e, e, ..., e,\ 1]$ respectively.

2. Preliminaries and remarks on the canonical module.

A fractional ideal of the value semigroup $\nu(R)$ is a subset $H \subseteq \mathbb{Z}$ such that $H + \nu(R) \subseteq H$. We denote by $c(H)$ the *conductor* of H, which is the smallest integer $j \in H$ such that $j + \mathbb{N} \subseteq H$. The number $\delta(H) := \#[\mathbb{N}_{\geq h_0} \setminus H]$ where $h_0 = min\{h \in H\}$ is the number of gaps of H. For any fractional ideal I of R, $\nu(I)$ is a fractional ideal of $\nu(R)$. Further we set:

$$c(I) := c(\nu(I)), \qquad \delta(I) := \delta(\nu(I)), \qquad c := c(R), \qquad \delta := \delta(R).$$

We point out the useful fact that, given two fractional ideals $I_1,\ I_2,\ I_2 \subseteq I_1$, the length of the R-module I_1/I_2 can be computed by means of valuations: $l_R(I_1/I_2) = \#[\nu(I_1) \setminus \nu(I_2)]$, (see [11], Proposition 1).

Now we collect some of the properties of the canonical module which are important in this context.

First we recall the following well-known:

Proposition 2.1. (see [8], [10], [12]) *Let ω be a canonical module of R such that $R \subseteq \omega \subseteq \overline{R}$ and let ω^{**} be its bidual, i.e. $\omega^{**} = R : (R : \omega)$. Then:*

1) $\omega : \omega = R$.
2) $l_R(I/J) = l_R(\omega : J/\omega : I)$.
3) $c(\omega) = c$ *and* $\nu(\omega) = \{j \in \mathbb{Z} | c - 1 - j \notin \nu(R)\}$.
4) $\omega : \overline{R} = \gamma$.
5) $\omega \subseteq \omega^{**} = \omega : \omega\theta_D$.
6) *R is Gorenstein* $\Longleftrightarrow \omega = R \Longleftrightarrow \theta_D = R \Longleftrightarrow \omega = \omega^{**}$. *Hence: R not Gorenstein* $\Longrightarrow\ \gamma \subseteq \theta_D \subseteq \mathfrak{m}$.
7) *If $S \supseteq R$ is an overring birational to R, then $\omega : S$ is a canonical module for S.*

Lemma 2.2. *Let I be a fractional ideal of R.*

i) *If $I \supseteq \gamma$ and $\nu(I) \subseteq \nu(\omega)$, then there exists a unit $u \in \overline{R}$ such that $uI \subseteq \omega$. If $\nu(I) = \nu(\omega)$, then $uI = \omega$.*
ii) *There exists a unit $u \in \overline{R}$ such that $ut^{c-c(I)}I \subseteq \omega$.*

Proof.

i) We note that $I \supseteq \gamma \Longrightarrow \omega : I \subseteq \overline{R} \Longrightarrow (\omega : I)\overline{R} \subseteq \overline{R}$. The hypotheses $I \supseteq \gamma$ and $\nu(I) \subseteq \nu(\omega)$ imply that $c(I) = c$, hence $I : \overline{R} = \gamma$ and $l_R(\overline{R}/(\omega : I)\overline{R}) = l_R(I : \overline{R}/\omega : \overline{R}) = 0$. From the equality $\overline{R} = (\omega : I)\overline{R}$ we deduce that $\omega : I$ contains a unit u of $\overline{R}$ and $uI \subseteq \omega$. The second assertion is now immediate, since $l_R(\omega/uI) = \#[\nu(\omega) \setminus \nu(I)] = 0$.
ii) We can apply item *i)* to the fractional ideal $t^{c-c(I)}I$, because the conditions $t^{c-c(I)}I \supseteq \gamma$ and $\nu(t^{c-c(I)}I) \subseteq \nu(\omega)$ are satisfied. □

A strict connection between the value sets of θ_D and ω^2 is remarked by D'Anna in [5], Lemma 3.2. Part iii) of next lemma is a slight generalization of it.

Lemma 2.3. *Let I be a fractional ideal of R. Let $h, s \in \mathbb{Z}$, $h \geq 1$. Then:*

i) $\nu(\omega : I) = \nu(\omega) - \nu(I)$.
ii) $\nu(\omega : I) = \{y \in \mathbb{Z} | c - 1 - y \notin \nu(I)\}$.
iii) $s \in \nu(R : \omega^{h-1}I) \Longleftrightarrow c - 1 - s \notin \nu(\omega^h I)$.

In particular: $s \in \nu(\theta_D) \Longleftrightarrow c - 1 - s \notin \nu(\omega^2)$.

Proof.

i) The proof given in [13], Prop. 2.4, works also under our assumptions.
ii) $\subseteq$ Using i), we see that $y \in \nu(\omega : I) \Longrightarrow c - 1 - y \notin \nu(I)$, since $c - 1 \notin \nu(\omega)$.
$\supseteq$ Let $y \in \mathbb{Z}$ be such that $c - 1 - y \notin \nu(I)$, and let $z \in \nu(I)$. Again by i) we can prove that $y + z \in \nu(\omega)$. Now $c - 1 - (y + z) = (c - 1 - y) - z \notin \nu(R) \Longrightarrow y + z \in \nu(\omega)$.
iii) Observe that $R : \omega^{h-1}I = \omega : \omega^h I$, then apply ii). □

Lemma 2.4. *Let I be a fractional ideal of R and let $J := I : \omega$. Then*

i) *J is a reflexive R-module, i.e. $J = R : (R : J)$.*
ii) *If J is not invertible, then $\mathfrak{m} : \mathfrak{m} \subseteq J : J$.*

In particular, θ_D is reflexive and $\mathfrak{m} : \mathfrak{m} \subseteq \theta_D : \theta_D$.

Proof.

i) The inclusion $J \subseteq R : (R : J)$ always holds. To prove $\supseteq$, observe that

$$x(R : J) \subseteq R \Longrightarrow x(R : J)\omega \subseteq \omega \Longrightarrow$$

$$x\omega \subseteq \omega : (R : J) = \omega : (\omega : J\omega) = J\omega \subseteq I \Longrightarrow x \in J.$$

ii) It suffices to note that

$$J \text{ not invertible } \Longrightarrow J(R : J) \neq R \Longrightarrow$$

$$J(R : J) \subseteq \mathfrak{m} \Longrightarrow J : J = R : J(R : J) \supseteq R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}. \qquad \square$$

In the last part of this section we point out how θ_D brings some relations with the bidual ω^{**} and the blowing up of the canonical module.

Denote by $B := \cup_{n=0,\dots,\infty} \omega^n : \omega^n$ the *blowing up of the canonical module* of R (independent on the choice of ω). This overring has been studied recently in relation to almost Gorenstein rings (see [2], ch.3, [5], ch.3).

Remark 2.5. *The ring B satisfies the following properties:*

i) *For $m >> 0$, $B = \omega^m : \omega^m = \omega^m$.* (See [5], 3).
ii) *B is a reflexive R-module.* In fact $B = (\omega^m : \omega^{m-1}) : \omega$ and we can apply Lemma 2.4.
iii) $\gamma \subseteq R : B \subseteq \theta_D$.
iv) $\omega(R : B) = \omega : B = R : B$. In fact $\omega(R : B) = \omega : (\omega : (\omega(R : B))) = \omega : B\omega : \omega^{m+1} = R : \omega^m = R : B$.
v) $\theta_D : \theta_D \subseteq B$. In fact $B = R : (R : B) = R : \omega(R : B) = \theta_D : (R : B) \supseteq \theta_D : \theta_D$.

Proposition 2.6. *The following facts hold:*

i) $\omega \subseteq \omega^{**} \subseteq \omega^2 \subseteq B \subseteq \overline{R}$.
ii) $l_R(\theta_D/\gamma) = l_R(\overline{R}/\omega^2)$.
iii) $l_R(\omega^2/\omega^{**}) = l_R(\omega\theta_D/\theta_D)$.
iv) *If R is not Gorenstein, then:*
$c(\omega^2) \leq c(\omega^{**}) \leq c - e$.
$c(\omega^2) = c - e \Longleftrightarrow e \in \nu(\theta_D)$.

Proof.

i) $\omega^{**} = R : (R : \omega) = \omega : \omega(\omega : \omega^2) \subseteq \omega : (\omega : \omega^2) = \omega^2$.

ii) Since $\omega : \gamma = \overline{R}$ and $\omega : \theta_D = \omega : (\omega : \omega^2) = \omega^2$, using the second property in Prop. 2.1, we get the thesis.

iii) is immediate by Prop. 2.1.

iv) $j \geq c-e \Longrightarrow c-1-j \leq e-1 \Longrightarrow$ either $c-1-j = 0$ or $c-1-j \notin \nu(R)$. Hence $j \in \nu(\omega) \cup \{c-1\} \subseteq \nu(\omega^{**})$.
Finally observe that $e \in \nu(\theta_D) \Longleftrightarrow c - 1 - e \notin \nu(\omega^2)$ by Lemma 2.3. □

Since a ring is Gorenstein if and only if $B = \omega$, it is now natural to set a characterization for the condition $B = \omega^2$. The condition is always verified by almost Gorenstein rings (see [2], Prop. 28). We point out that there exist not almost Gorenstein rings with $B = \omega^2$, for instance the semigroup ring $R = \mathbb{C}[[t^h]], h \in \nu(R) = \{0, 7, 8, 9, 11, 13, \rightarrow\}$.

Theorem 2.7. *The following conditions are equivalent:*

i) ω^{**} *is a ring.*

ii) $\omega^{**} = \omega^2$.

iii) $\omega\theta_D = \theta_D$.

iv) $\theta_D : \theta_D = B$.

v) $R : B = \theta_D$.

vi) $B = \omega^2$.

Proof.

i) $\Longrightarrow ii)$. In this hypothesis: $\omega \subseteq \omega^{**} \subseteq \omega^2 \subseteq \omega\omega^{**} = \omega^{**}$.

ii) $\Longrightarrow iii)$ is immediate by Prop. 2.6.

iii) $\Longrightarrow iv)$ $\omega\theta_D = \theta_D \Longrightarrow \omega^m\theta_D = \theta_D \Longrightarrow B \subseteq \theta_D : \theta_D$ and the other inclusion always holds (see Remark 2.5).

iv) $\Longrightarrow v)$ $\theta_D : \theta_D = B \Longrightarrow B\theta_D \subseteq R \Longrightarrow \theta_D \subseteq R : B$ and the other inclusion always holds (see Remark 2.5).

v) $\Longrightarrow vi)$ $\theta_D = \omega : \omega^2 = R : B = \omega : B\omega = \omega : B \Longrightarrow \omega : (\omega : \omega^2) = \omega : (\omega : B)$.

vi) $\Longrightarrow i)$ $\omega^3\theta_D = \omega^2\theta_2 \subseteq \omega \Longrightarrow \omega^2 \subseteq \omega : \omega\theta_D = \omega^{**} \Longrightarrow \omega^{**} = B$. □

3. Type-sequences and length.

The number p of 1's in $t.s.(R)$, is related to the length of the $R/\mathfrak{m}$-algebra R/θ_D and is involved in other interesting inequalities. First we show (Prop. 3.4) how elements of $\nu(\theta_D)$ give rise to 1's in $t.s.(R)$, and in $t.s.(\theta_D)$. From this we get $\delta \leq (c-\delta)r_1 - p(r_1 - 1) \leq (c-\delta)r_1 - l_R(\theta_D/\gamma(r_1 - 1)$ (Theorem 3.7) and we state other bounds.

Proposition 3.1. (see [5]) *Let* $\nu(R) = \{s_0 = 0, s_1, ...s_n = c, \rightarrow\}$, $n = c - \delta$, *and let* $t.s.(R) = [r_1, ..., r_n]$ *and* $t.s.(\theta_D) = [t_1, ..., t_n]$ *be the type sequences of* R *and* θ_D *respectively. Then:*

i) $c(\theta_D : R_i) = c(R : R_i) = c - s_i$, *for each* $i = 0, ..., n$.

ii) $\nu(\theta_D : R_i)_{<c-s_i} = \{c - 1 - b,\ b \in \mathbb{Z}_{\geq s_i} \setminus \nu(\omega^2 R_i)\}$, *for each* $i = 0, ..., n$.

iii) Let $n_i := c(R : R_i) - \delta(R : R_i)$, $m_i := c(\theta_D : R_i) - l_R(\overline{R}/\theta_D : R_i)$. *Then:*

1. $r_{i+1} = s_{i+1} - s_i + n_{i+1} - n_i, \quad i = 0, ..., n - 1.$
2. $t_{i+1} = s_{i+1} - s_i + m_{i+1} - m_i, \quad i = 0, ..., n - 1.$
3. $\sum_{i=1}^{n} r_i = \delta.$
4. $\sum_{i=1}^{n} t_i = \delta + l_R(R/\theta_D).$

iv) Denoting by ω_i *the canonical module* $\omega : (R : R_i)$ *of the overring* $R : R_i$ *obtained by duality, we have:* $r_i = l_R(\omega_{i-1}/\omega_i)$.

Proof. By Lemma 2.3 we have that: $x \in \nu(\theta_D : R_i) \iff c - 1 - x \notin \nu(\omega^2 R_i)$.

i) If $j \geq c - s_i \implies c - 1 - j < s_i \implies c - 1 - j \notin \nu(\omega^2 R_i) \implies j \in \nu(\theta_D : R_i) \subseteq \nu(R : R_i)$. Moreover $s_i \in \nu(\omega R_i) \implies c - s_i - 1 \notin \nu(R : R_i)$ by Lemma 2.3.

ii) follows from the above considerations.

iii) For the first equality see [5]. The second one is analogous: by definition and item i), $m_{i+1} = c - s_{i+1} + l_R(\overline{R}/\theta_D : R_{i+1})$ and $m_i = c - s_i + l_R(\overline{R}/\theta_D : R_i)$. Since $l_R(\overline{R}/\theta_D : R_i) - l_R(\overline{R}/\theta_D : R_{i+1}) = l_R(\theta_D : R_{i+1}/\theta_D : R_i) = t_{i+1}$, we get the thesis by subtraction. The other equalities are immediate by definition.

iv) Apply Prop. 2.1, 7): $\omega_i = \omega : (R : R_i) = \omega : (\omega : \omega R_i) = \omega R_i$. $\square$

Proposition 3.2. *Let* $t.s.(R) = [r_1, ..., r_n]$ *and* $t.s.(\theta_D) = [t_1, ..., t_n]$. *Let* $x_{i-1} \in \mathfrak{m}$ *be such that* $\nu(x_{i-1}) = s_{i-1} < c$. *Then:*

i) $r_i = 1 \iff x_{i-1} \in Ann_R\big(\omega/(x_{i-1}R + \omega R_i)\big)$.

ii) $r_i = 1 \implies t_i = 1$.

Proof.

i) Since $R_{i-1} = x_{i-1}R + R_i$, we have $\omega R_{i-1} = x_{i-1}\omega + \omega R_i$. Then $r_i = l_R(\omega R_{i-1}/\omega R_i) = 1 \iff \omega R_{i-1} = x_{i-1}R + \omega R_i \iff x_{i-1} \in Ann_R\big(\omega/(x_{i-1}R + \omega R_i)\big)$.

ii) By hypothesis $\omega R_{i-1} = x_{i-1}R + \omega R_i \implies \omega^2 R_{i-1} = x_{i-1}\omega + \omega^2 R_i$, hence by i), $\omega^2 R_{i-1} = x_{i-1}R + \omega^2 R_i \implies t_i = l_R(\omega^2 R_{i-1}/\omega^2 R_i) = 1$. $\square$

Lemma 3.3. ([5], Lemma 4.1) *Let* $z_1, ..., z_r$ *be any minimal set of generators of* ω. *Then, if* $x_i \in R$ *and* $\nu(x_i) = s_i$, *the* R*-module* $\omega R_i/\omega R_{i+1}$ *is generated by* $x_i z_1 + \omega R_{i+1}, ..., x_i z_r + \omega R_{i+1}$.

Proposition 3.4. *Let $t.s.(R) = [r_1, ..., r_n]$ and $t.s.(\theta_D) = [t_1, ..., t_n]$ be the type sequences of R and θ_D respectively. Then :*

$$s_i \in \nu(\theta_D) \Longrightarrow r_{i+1} = t_{i+1} = 1.$$

Proof. $r_{i+1} = l_R(\omega R_i/\omega R_{i+1})$. Let $\omega = (1, z_2, ..., z_r)$ and let $x_i \in \theta_D$ be such that $\nu(x_i) = s_i < c$. Then $\omega R_i =< x_i, ..., x_i z_r >$ mod ωR_{i+1}, by Lemma 3.3. Thus $x_i \in R : \omega \Longrightarrow x_i z_j \in R_{i+1} \subseteq \omega R_{i+1}$ for all $j > 1$ (since $\nu(x_i z_j) > i) \Longrightarrow r_{i+1} = 1$ and by Prop. 3.2, $t_{i+1} = 1$. □

Notation 3.5. *We put:*

$$p := \# [i \in \{1, ..., c - \delta\} \mid r_i = 1]$$

$$\sigma := l_R(\omega/R) - l_R(R/\theta_D) = 2\delta - c - l_R(R/\theta_D)$$

The invariant σ has been introduced in [9]. It is known that $\sigma(R) \geq 0$, when $r_1 \leq 3$ or R is smoothable, but there are examples with $\sigma < 0$ (see 4.12).

Lemma 3.6. *The following facts hold:*

i) $l_R(\theta_D/\gamma) \leq p$.
ii) $c - \delta - p \leq l_R(R/\theta_D) \leq c - \delta$.
iii) $3\delta - 2c \leq \sigma \leq 3\delta - 2c + p$.
iv) $c - p \leq \sum_{i=1}^{n} t_i \leq c$.

Proof.

i) follows from Prop. 3.4.
ii) First inequality comes from i), since $l_R(R/\theta_D) = l_R(R/\gamma) - l_R(\theta_D/\gamma)$; the second one holds since $\gamma \subseteq \theta_D$.
iii) is obvious by ii).
iv) $l_R(R/\theta_D) + \delta = \sum_{i=1}^{n} t_i$, so the inequalities are immediate from ii). □

Theorem 3.7. *Let p be the number defined in 3.5. Then:*

$$2(c-\delta) - p \leq \delta \leq (c-\delta)r_1 - p(r_1 - 1) \leq (c-\delta)r_1 - l_R(\theta_D/\gamma)(r_1 - 1).$$

Proof. Since $r_{i_1} = \ldots = r_{i_p} = 1$, and $r_i \leq r_1 \forall i$, using Prop. 3.1, iii) we get:

$$c-\delta+(c-\delta-p) \leq \delta = \sum_{1}^{c-\delta} r_i = c-\delta+\sum_{1}^{c-\delta}(r_i-1) \leq c-\delta+(c-\delta-p)(r_1-1).$$

To get the last inequality use Lemma 3.6, i). □

Corollary 3.8. *Let, as above, $n = c - \delta$. Then:*

i) $2\delta - c = \sum_{i=1}^{n}(r_i - 1) \leq (c - \delta - p)(r_1 - 1) \leq l_R(R/\theta_D)(r_1 - 1)$.
ii) $2\delta - c \leq l_R(R/\theta_D)(t_1 - 2)$.

Proof.

i) See the proof of Theorem 3.7, then use Lemma 3.6, ii).
ii) As in the proof of Theorem 3.7, using Prop. 3.1 and Prop. 3.2, we obtain:

$$2\delta - c + l_R(R/\theta_D) = \sum_{i=1}^{n}(t_i - 1) \leq (c - \delta - p)(t_1 - 1) \leq l_R(R/\theta_D)(t_1 - 1).$$

□

Corollary 3.9. *Either $t_1 = 1$ (i.e. R is Gorenstein) or $t_1 \geq 3$.*

From the first inequality of Theorem 3.7 we deduce the following

Corollary 3.10. $p \geq 2c - 3\delta$.

Of course, the above lower bound for p is significant in the case $2c - 3\delta > 0$. Using iii) of Lemma 3.6 we see that if $\sigma < 0$, then $2c - 3\delta > 0$. Example 5 in 4.12 shows that the converse is false. The following bound for $l_R(R/\theta_D)$ is non trivial when $\sigma < 0$ (see Example 4 in 4.12).

Proposition 3.11. $l_R(R/\theta_D) \leq (2\delta - c)(r_1 - 1)$.

Proof. Let $\omega = (1, z_2, \ldots, z_{r_1})R$ and consider, as in [10], Satz 3), for every $i = 1, \ldots, r_1$ the R-module $\omega_i := (1, \ldots, z_i)R$. In particular ω_2 is two-generated, so by [3], Satz 2, $l_R(R/R : \omega_2) = l_R(\omega_2/R)$. It is clear that $\omega_{i+1}/\omega_i \simeq R/\mathfrak{b}_{i+1}$, where $\mathfrak{b}_{i+1} = Ann_R(\omega_{i+1}/\omega_i)$. By [10], Hilfssatz 4 and Satz 1 we obtain: $l_R(R : \omega_i/R : \omega_{i+1}) \leq l_R(R : \mathfrak{b}_{i+1}/R) \leq l_R(R/\mathfrak{b}_{i+1}) + 2\delta - c = l_R(\omega_{i+1}/\omega_i) + 2\delta - c$. Since $R = R : \omega_1 \supset R : \omega_2 \supset .... \supset R : \omega_{r_1} = \theta_D$, we have $l_R(R/\theta_D) = l_R(R/R : \omega_2) + \sum_{i=2}^{r_1-1} l_R(R : \omega_i/R : \omega_{i+1}) \leq l_R(\omega_2/R) + \sum_{i=2}^{r_1-1} l_R(\omega_{i+1}/\omega_i) + (2\delta - c)(r_1 - 2) = l_R(\omega/R) + (2\delta - c)(r_1 - 2)$. The thesis follows. □

Remark 3.12. The difference $a := (c - \delta)r_1 - \delta$ has been taken into account by several authors. In [10] it is proved that $a \geq 0$, when R is a one-dimensional local analytically unramified Cohen Macaulay ring. In [11] it had already been shown that $a \geq 0$, under more particular hypotheses. In [4] some general stucture theorems are presented for rings with $a = 0$ (the so called rings *of maximal length*) or $a = 1$ (the so called rings *of almost maximal length*).

Theorem 3.7 implies that $a \geq l_R(\theta_D/\gamma)(r_1 - 1)$. Hence:

$a < r_1 - 1 \Longrightarrow \theta_D = \gamma$.
$a = r_1 - 1 \Longrightarrow l_R(\theta_D/\gamma) \leq 1$.

The cases $a \leq r_1 - 1$ are studied in [6] and [7]. See also the following 5.2.

4. Relations between r_i's and t_i's.

Starting from the almost Gorenstein case, we are led to consider in a $t.s.$ $[r_1, \dots, r_i, 1, 1, \dots, 1]$ the index i of the last element r_i which is not 1. This number has a central role in Theorem 4.3 which involves R_i, θ_D and B. When $i = 1$, this theorem gives again the known characterizations of almost Gorenstein rings.

Lemma 4.1. *Let J be any proper ideal of R. If $\nu(R_i) \subseteq \nu(J)$, then $R_i \subseteq J$.*

Proof. In fact

$$\nu(R_i) \subseteq \nu(J) \Longrightarrow \nu(R_i \cap J) = \nu(R_i) \Longrightarrow R_i \cap J = R_i \Longrightarrow R_i \subseteq J. \ \square$$

Lemma 4.2. *The following facts hold:*

i) $r_{i+1} > 1 \Longrightarrow c - 1 \in \nu(\omega^2 R_i)$.
ii) $c - 1 \in \nu(\omega^2 R_i) \Longleftrightarrow R_i \not\subseteq \theta_D$.
iii) If $r_n > 1$, *then* $t_n \geq r_n + 1$.

Proof.

i) By Prop. 3.4, $r_{i+1} > 1 \Longrightarrow s_i \notin \nu(\theta_D) \Longrightarrow c - 1 - s_i \in \nu(\omega^2) \setminus \nu(\omega) \Longrightarrow c - 1 = s_i + (c - 1 - s_i) \in \nu(\omega^2 R_i)$.

ii) By Lemma 2.3 $c - 1 \in \nu(\omega^2 R_i) \Longleftrightarrow 0 \notin \nu(R : \omega R_i)$. Suppose $c - 1 \in \nu(\omega^2 R_i)$. If $R_i \subseteq \theta_D$, then $1 \in \theta_D : R_i = R : \omega R_i$, contradiction. Vice versa, if $R_i \not\subseteq \theta_D$, by Lemma 4.1 there exists an element $x \in R_i \setminus \theta_D$ such that $\nu(x) \notin \nu(\theta_D)$; then $u\,x\omega \not\subseteq R$ for all units $u \in \overline{R}$. It follows that $0 \notin \nu(R : \omega R_i)$.

iii) We have: $r_n = l_R(\omega R_{n-1}/\omega R_n) = l_R(\omega R_{n-1}/\gamma) \leq l_R(\omega^2 R_{n-1}/\gamma) = l_R(\omega^2 R_{n-1}/\omega^2 R_n) = t_n$. Looking at valuations we see that the above inequality is strict since $c - 1 \in \nu(\omega^2 R_{n-1}) \setminus \nu(\omega R_{n-1})$, by i). $\square$

In [2] it is proved that

$$R \text{ is almost Gorenstein} \Longleftrightarrow \mathfrak{m} = \omega\,\mathfrak{m} \Longleftrightarrow r_1 - 1 = 2\delta - c.$$

Hence: R almost Gorenstein, not Gorenstein$\Longleftrightarrow \theta_D = \mathfrak{m}$. In other words:
$t.s.(R) = [r_1, \dots, 1]$ with $r_1 > 1 \Longleftrightarrow R_1 \subseteq \theta_D$ and $R_0 \not\subseteq \theta_D$.
Next proposition is a generalization of this fact.

Theorem 4.3. *Let $1 \leq i \leq n$ and let $B = \omega^m$ be the blowing up of the canonical module of R. The following are equivalent:*

i) $R_i \subseteq \theta_D$ *and* $R_{i-1} \nsubseteq \theta_D$.
ii) $B \subseteq R : R_i$ *and* $B \nsubseteq R : R_{i-1}$.
iii) $t.s.(R) = [r_1, \dots, r_i, 1, 1, \dots, 1]$ *with* $r_i > 1$.
iv) $t.s.(\theta_D) = [t_1, \dots, t_i, 1, 1, \dots, 1]$ *with* $t_i > 1$.

Proof.

i) $\Longleftrightarrow$ ii) $R_i \subseteq \theta_D \Longleftrightarrow \omega R_i = R_i \Longleftrightarrow \omega^m R_i = R_i \Longleftrightarrow B \subseteq R : R_i$.

i) $\Longrightarrow$ iii) By hypothesis $s_j \in \nu(\theta_D)\ \forall j \geq i \Longrightarrow r_j = 1\ \forall j > i$. We have to prove that $r_i > 1$. If $r_i = 1$, then by Prop. 3.2, i), $\omega R_{i-1} = x_{i-1}R + \omega R_i \subseteq R \Longrightarrow R_{i-1} \subseteq \theta_D$, absurd.

iii) $\Longrightarrow$ iv) $r_i = l_R(\overline{R}/R : R_{i-1}) - l_R(\overline{R}/R : R_i) = l_R(\overline{R}/R : R_{i-1}) - (n-i)$ and analogously, by Prop. 3.2, ii), $t_i = l_R(\overline{R}/\theta_D : R_{i-1}) - (n-i) \Longrightarrow t_i \geq r_i > 1$.

iv) $\Longrightarrow$ iii) If $i = n$, the implication is true by Prop. 3.2, ii). Let $i \leq n-1$. Surely, by Prop. 3.2, $r_i > 1$ and by Lemma 4.2, iii), $r_n = 1$. If $r_j > 1$ with $i < j < n$ and all the subsequents equal to 1, as above we would get $t_j \geq r_j > 1$, contradiction.

iii) $\Longrightarrow$ i) $r_n = 1 \Longrightarrow \omega R_{n-1} = x_{n-1}R + \gamma \subseteq R \Longrightarrow R_{n-1} \subseteq \theta_D$. If also $r_{n-1} = 1$, then $\omega R_{n-2} = x_{n-2}R + \omega R_{n-1} \subseteq R$, then $R_{n-2} \subseteq \theta_D$ and so on. If $R_{i-1} \subseteq \theta_D$, then $r_i = 1$, and this concludes the proof. □

Proposition 4.4. *If $i \leq n$ is such that $r_i > 1$ and $r_j = 1$ for all $j \geq i+1$,*

$$\textit{then} \qquad t_i = r_i + 1.$$

In particular: $r_n > 1 \Longrightarrow t_n = r_n + 1$.

Proof. By Theorem 4.3 we have $R_i \subseteq \theta_D$, hence $r_i = l_R(\omega R_{i-1}/R_i)$ and $t_i = l_R(\omega^2 R_{i-1}/R_i)$. Since, by Lemma 4.2, i), $c-1 \in \nu(\omega^2 R_{i-1})$, our thesis will follow by proving that $\nu(\omega^2 R_{i-1}) = \nu(\omega R_{i-1}) \cup \{c-1\}$. Hence, let $m \in \nu(\omega^2 R_{i-1}) \setminus \nu(\omega R_{i-1})$: we claim that $m = c-1$. By Lemma 2.3 $c-1-m \in \nu(R : R_{i-1})$. Let $m = \nu(x),\ x \in \omega^2 R_{i-1}$ and $c-1-m = \nu(y),\ y \in R : R_{i-1}$. If $\nu(y) > 0$, then $yR_{i-1} \subseteq R_i$, hence $c-1 = \nu(xy) \in \nu(\omega^2 R_i) = \nu(R_i)$, absurd. Hence $\nu(y) = 0$ and the thesis is achieved. □

Proposition 4.5. *The following are equivalent:*

i) $s_{n-1} \in \nu(\theta_D)$.
ii) $s_{n-1} = c-2$.
iii) $r_n = 1$.

Proof. Recall that $\omega R_n = \gamma$.

i) $\Longrightarrow$ ii). If $c-2 \notin \nu(R)$, then $1 \in \nu(\omega)$. But this would imply that s_{n-1} and $s_{n-1}+1 \in \nu(\omega R_{n-1}) \setminus \nu(\gamma) \Longrightarrow r_n > 1 \Longrightarrow s_{n-1} \notin \nu(\theta_D)$, absurd.

ii) $\Longrightarrow$ iii) Obviously $\nu(\omega R_{n-1}) \setminus \nu(\gamma) = \{s_{n-1}\}$. □

Corollary 4.6. $B = \overline{R} \Longleftrightarrow r_n > 1$.

Proof. $B = \overline{R} \Longleftrightarrow 1 \in \nu(\omega) \Longleftrightarrow c-2 \notin \nu(R)$. □

Corollary 4.7. *If $\theta_D = R_i$ for some i, then the equivalent conditions of Theorem 2.7 hold.*

Proof. $B \subseteq R : R_i$ by Theorem 4.3 $\Longrightarrow R : B \supseteq R_i = \theta_D \Longrightarrow R : B = \theta_D$, since the other inclusion is always true. □

In the particular case $\theta_D = R_n$ we obtain:

Proposition 4.8. *Set, as above, $n_i := c(R : R_i) - \delta(R : R_i)$ and $m_i := c(\theta_D : R_i) - l_R(\overline{R}/\theta_D : R_i)$. The following facts are equivalent:*

i) $\theta_D = \gamma$.
ii) $\omega^2 = \overline{R}$.
iii) $t_i = s_i - s_{i-1}$ *for each* $i = 1, ..., n$.
iv) $m_i = 0$ *for each* $i = 0, ..., n$.
v) $\theta_D : R_i = t^{c-s_i}\overline{R}$ *for each* $i = 0, ..., n$.
vi) $\omega^{**} = \overline{R}$.

If the above conditions hold, then

a) $t_1 = e$.
b) $\forall\ i > 1,\quad r_i > t_i \Longleftrightarrow n_i > n_{i-1}$.

Proof.

i) $\Longleftrightarrow$ ii) See Prop. 2.6, *ii*).
ii) $\Longrightarrow$ iii) In fact $t_i = l_R(\omega^2 R_i/\omega^2 R_{i-1}) = l_R(R_i\overline{R}/R_{i-1}\overline{R}) = s_i - s_{i-1}$.
iii) $\Longrightarrow$ iv) We have seen in Prop. 3.1 that $t_i = s_i - s_{i-1} + m_i - m_{i-1}$. Hypothesis *iii*) implies that $m_1 = m_2 = ... = m_n = c(\overline{R}) - \delta(\overline{R}) = 0$.
iv) $\Longrightarrow$ v) $m_i = 0 \Longrightarrow \nu(\theta_D : R_i) = [c - s_i, +\infty)$. Since the inclusion $t^{c-s_i}\overline{R} \subseteq \theta_D : R_i$ holds for every $i = 0, ..., n$, the equality of the value sets implies the other inclusion.
v) $\Longrightarrow$ i) Take in v) $i = 0$.
vi) $\Longrightarrow$ ii) and *i*) $\Longrightarrow$ *vi*) are immediate by Prop. 2.6.
a) $t_1 = s_1 - s_0 = e$.
b) Using Prop. 3.1 iii), it is immediate. □

Our conjecture $t_1 \geq r_1$ is true for rings having maximal C.M. type, namely $r_1 = e - 1$. In this case we get a more precise result.

Proposition 4.9. *Let $e \geq 3$. If for some $1 \leq i \leq n$ $r_i = e-1$, then $t_i = e$. Moreover, for the same i we have: $s_{i-1} = (i-1)e, \quad s_i = ie$.*

Proof. Since $t^e R_{i-1} \subseteq R_i \subset R_{i-1}$, we have the chain $t^e \omega R_{i-1} \subseteq \omega R_i \subseteq \omega R_{i-1}$. Hypothesis $r_i = e-1$ implies that $l_R(\omega R_i / t^e \omega R_{i-1}) = 1$ and since $c-1+e \in \nu(\omega R_i) \setminus \nu(t^e \omega R_{i-1})$, it follows that

$$(*) \qquad \omega R_i = t^e \omega R_{i-1} + zR \ \text{ with } \ \nu(z) = c-1+e.$$

Analogously, considering the chain $t^e \omega^2 R_{i-1} \subseteq \omega^2 R_i \subseteq \omega^2 R_{i-1}$, we see that the thesis $t_i = e$ is equivalent to $t^e \omega^2 R_{i-1} = \omega^2 R_i$. It will be sufficient to prove this last equality. From $(*)$ we have $\omega^2 R_i = t^e \omega^2 R_{i-1} + z\omega$. Now, $z \in \gamma \subseteq R_i$ for every $i \Longrightarrow z\omega \subseteq \omega R_i \Longrightarrow \omega^2 R_i = t^e \omega^2 R_{i-1} + zR$. By Lemma 4.2 $r_i > 1 \Longrightarrow c-1 \in \nu(\omega^2 R_{i-1})$, then $\nu(z) \in \nu(t^e \omega^2 R_{i-1})$: we obtain that $t^e \omega^2 R_{i-1} = \omega^2 R_i$, as claimed.

To prove the other equalities, note that by definition $s_i \leq s_{i-1} + e$. As already remarked $r_i = e-1$ implies that $\nu(\omega R_i) = \nu(t^e \omega R_{i-1}) \cup \{c-1+e\}$. Hence $s_i \in \nu(t^e \omega R_{i-1})$, but $s_i \geq s_{i-1} + e \Longrightarrow s_i = s_{i-1} + e = ie$. □

For rings of C.M. type 2, we have a complete description of the type sequences of R and θ_D. In this case the arrow $\Longrightarrow$ of Prop. 3.4 becomes $\Longleftrightarrow$.

Proposition 4.10. *Suppose $r_1 = 2$. Then:*

$$s_i \in \nu(\theta_D) \Longrightarrow r_{i+1} = t_{i+1} = 1$$
$$s_i \notin \nu(\theta_D) \Longrightarrow r_{i+1} = 2, \ t_{i+1} = 3.$$

Proof. We have from Corollary 3.8, i) and Prop. 3.11 that $l_R(R/\theta_D) = 2\delta - c$ hence $l_R(\theta_D/\gamma) = 2c - 3\delta$. The elements of the type sequence $[r_1, ..., r_n]$, $n = c-\delta$, of R are 1 or 2, suppose p times 1 and $n-p$ times 2. Then $\delta = \sum_{i=1}^n r_i = p + 2(n-p) \Longrightarrow p = 2c - 3\delta$. Hence $p = l_R(\theta_D/\gamma)$ and $r_{i+1} = 1 \Longleftrightarrow s_i \in \theta_D$ (see Prop. 3.4). By hypothesis ω is two-generated, say $\omega = (1, z)$, then $1, z, z^2$ constitute a system of generators for ω^2; hence $t_1 \leq 3$, and Corollary 3.9 implies that $t_1 = 3$. Consider now the type sequence of θ_D, by Prop. 3.2, $r_i = 1 \Longrightarrow t_i = 1$. Suppose that for some i either $t_i = 2$ or $r_i = 2$ and $t_i = 1$. Then $\delta + l_R(R/\theta_D) = \sum_{i=1}^n t_i < l_R(\theta_D/\gamma) + 3 l_R(R/\theta_D) \Longrightarrow \delta < c - \delta + 2\delta - c$, absurd. The thesis follows. □

Another case in which our conjecture $t_1 \geq r_1$ is true comes directly from Corollary 3.8:

Proposition 4.11. *If $l_R(R/\theta_D)(r_1 - 2) \leq 2\delta - c$, then $r_1 \leq t_1$.*

Proof. If $r_1 > t_1$, from Corollary 3.8, ii), we get $2\delta - c \leq l_R(R/\theta_D)(t_1 - 2) < l_R(R/\theta_D)(r_1 - 2)$. $\square$

Example 4.12. Suppose $R = \mathbb{C}[[t^h]]$, $h \in \nu(R)$, is a semigroup ring. The first three examples show that the converses of Prop. 3.2, ii), Prop. 3.4 and Prop. 4.9 are false.

1. Let $\nu(R) = \{0, 10, 11, 17, 20 \rightarrow\}$, then $\theta_D = \gamma$, $\delta = 16$, $c - \delta = 4 < 12 = 2\delta - c$, $t.s.(R) = [7, 2, 5, 2]$, $t.s.(\theta_D) = [10, 1, 6, 3]$. In this case $t_2 = 1$ and $r_2 > 1$.
2. Let $\nu(R) = \{0, 5, 6, 10 \rightarrow\}$, then $\theta_D = \gamma$, $\delta = 7$, $c - \delta = 3 < 4 = 2\delta - c$, $t.s.(R) = [3, 1, 3]$, $t.s.(\theta_D) = [5, 1, 4]$. In this case $t_2 = r_2 = 1$. But $s_1 = 5 \notin \nu(\theta_D)$.
3. Let $\nu(R) = \{0, 10, 11, 12, 14, 17, 20 \rightarrow\}$. Then: $c = 20$, $\delta = 14$, $r_1 = 5$, $\omega = \langle 0, 1, 3, 4, 6 \rangle$, $\omega^2 = \overline{R}$, hence $\theta_D = \gamma$. $t.s(R) = [5, 1, 1, 3, 2, 2]$, $t.s.(\theta_D) = [10, 1, 1, 2, 3, 3]$. In this case $t_1 = 10$, but $r_1 = 5 < e - 1$, moreover $r_4 > t_4 = 2$.
4. Let $\nu(r) = \langle 13, 121, 133, 163, 164, 166, 168, 170, 171 \rangle$. We have $\delta = 181$, $c = 322$, $r_1 = 4$, $\theta_D = \langle 121, 166, 168, 198, 216, 223, 234, 241, 248, 266 \rangle$. Hence $l_R(R/\theta_D) = 43$ and $\sigma = -3$. Here bound in Prop. 3.11 is better than bound in Lemma 3.6, ii). In fact: $2\delta - c = 40 < l_R(R/\theta_D) = 43 < (2\delta - c)(r_1 - 1) = 120 < c - \delta = 141$. The type sequences $t.s.(R)$ and $t.s.(\theta_D)$ are respectively:
   [4 4 4 4 4 3 2 2 2 2 1 2 2 1 2 1 1 1 1 1 2 1 1 1 1 2 2 1 1 2 1 1 1 1 2 2 1 1 2 1 1
   1 1 2 2 1 1 2 1 1 1 1 2 2 1 1 2 1 1 1 1 2 1 1 1 2 1 1 1 1 1 2 1 ... 1]
   [10 10 10 10 8 6 3 3 3 3 1 3 2 1 3 1 1 1 1 1 2 1 1 1 1 3 2 1 1 2 1 1 1 1 3 2 1 1 2 1 1
   1 1 3 2 1 1 2 1 1 1 1 3 2 1 1 2 1 1 1 1 3 1 1 1 2 1 1 1 1 1 3 1 ... 1]

5 Let $\nu(R) = \{7, 8, 9, 10, 12 \rightarrow\}$. We have $\delta = 7$, $r_1 = 3$, $c = 12$. and R is almost Gorenstein, so $\theta_D = \mathfrak{m}$, hence $\sigma = 1$, but $3\delta - 2c < 0$.

5. Minimality and maximality.

In the comparison between the type sequences of the ring and of the Dedekind different, properties like minimality and maximality are completely equivalent.

• **Minimal type sequences .** In [2] one can find the properties of *almost Gorenstein* rings. Analogous properties for fractional ideals are considered in [13]: a fractional ideal I is called of *minimal type sequence* (*m.t.s.* for short) if and only if $t.s.(I) = [r(I), 1, ..., 1]$, where $r(I)$ is the Cohen Macaulay type of I as an R-module. Since it is well known that $r(I) = 1 \iff I \simeq \omega$, it follows in particular that $t_1 = 1 \implies R$ is Gorenstein.

Next proposition deals with the m.t.s. property in the not Gorenstein case.

Proposition 5.1. *Let R be not Gorenstein. The following are equivalent:*

i) *R is almost Gorenstein.*
ii) *θ_D is m.t.s.*
iii) *$\omega^{**} = R : \mathfrak{m}$,*
iv) *$B = R : \mathfrak{m}$.*

In this case $t_1 = r_1 + 1$.

Proof.

i) $\iff$ ii) is equivalence iii) $\iff$ iv) of Theorem 4.3 for $i = 1$.
i) $\implies$ iii) is immediate, since when R is almost Gorenstein, we have $\theta_D = \mathfrak{m} = \mathfrak{m}\omega$ and by Prop. 2.6 $\omega^{**} = \omega^2 = R : \mathfrak{m}$. Last equality is proved in [2], Prop. 28.
iii) $\implies$ iv) ω^{**} is a ring $\implies \omega^{**} = \omega^2 = B$ by Theorem 2.7.
i) $\implies$ iv) has been proved by D'Anna in [5], Prop. 3.4. □

• **Maximal type sequences.** Recalling that in general $t.s.(R) = [r_1, ..., r_n]$, with $r_1 \leq e - 1$ and $r_i \leq r_1$, of course "maximal" type sequence means $t.s.(R) = [e-1, ..., e-1]$. In [7] and [6] the authors characterize all the rings whose type sequence is closer to the maximal one in the following sense: $t.s.(R) = [e-1, ..., e-1, e-1-a]$. For simplicity, we call *a-maximal* a type sequence of this form.

Proposition 5.2. *(See [6] and [7]). Let $a \in \mathbb{N}$ be such that $a \leq r_1 - 1$. The following facts are equivalent:*

i) *$(c - \delta)r_1(R) - \delta = a$ and $r_1 = e - 1$.*
ii) *$\nu(R) = \{0, e, 2e, ..., (n-1)e, ne - a, \rightarrow\}$.*
iii) *$t.s.(R) = [e - 1, ..., e - 1, e - 1 - a]$.*

Moreover, if $a \leq r_1 - 2$, then condition $r_1 = e - 1$ in i) is superflous.

We want to show now that the a-maximality of $t.s.(R)$ is equivalent to the a-maximality of $t.s.(\theta_D)$, i.e. $t.s.(\theta_D) = [e, ..., e, e - a]$, (see Prop. 5.4). To do this we need some more or less well known results, that we list below for our convenience. In the following $\langle l_1, ..., l_i\rangle$ denotes the $\nu(R)$-set generated by $l_1, ..., l_i$ and, for any numerical set $H \subset \mathbb{Z}$, $H + l := \{h + l,\ h \in H\}$.

Lemma 5.3. *Let* $0 \le a \le e-2$ *and let* $\nu(R) = \{0, e, 2e, ..., (n-1)e, ne - a, \rightarrow\}$. *In this case* $c = ne - a,\ n = c - \delta$.

i) Canonical ideals:
For $a = 0$ *then* $\nu(\omega) = \langle 0, 1, 2, ..., e-2\rangle$. *Call it* $\nu(\omega_0)$.
For any $a \ge 1$, *change the last* a *generators by addying* 1 *to each one, i.e.* $\nu(\omega_a) = \langle 0, 1, ..., e-a-2, e-a, ..., e-1\rangle$.
In particular, $\nu(\omega_{e-2}) = \langle 0, 2, 3, ..., e-1\rangle$.

ii) Type sequence of R :
$t.s.(R) = [e-1, ..., e-1, e-1-a]$.

iii) Omega square:
for $a = 0, ..., e-3$ $\quad \omega^2 = \overline{R}$,
for $a = e-2$ $\quad \nu(\omega^2) = \{0, 2, \rightarrow\}$.

iv) Type sequence of θ_D :
for $a = 0, ..., e-3$ $\quad t.s.(\theta_D) = [e, e, ..., e, e-a]$,
for $a = e-2$ $\quad t.s.(\theta_D) = [e, e, ..., e,\ 1]$.

v) Dedekind different:
for $a = 0, ..., e-3$ $\quad \theta_D = \gamma$,
for $a = e-2$ $\quad \theta_D = zR + \gamma$ *with* $\nu(z) = (n-1)e$.

Proof.

i) Just remember that $\nu(\omega) = \{j \in \mathbb{Z} \mid c - 1 - j \notin \nu(R)\}$.

ii) For every $a = 0, ..., e-2$ and for every $i = 0, ..., n-1$, we have $\nu(\omega R_i) = \nu(\omega) + ie$. Then for every $i = 0, ..., n-2$, $\nu(\omega R_i) \setminus \nu(\omega R_{i+1}) = \{0, 1, ..., e-a-2, e-a, ..., e-1\} + ie$. So we obtain that $r_{i+1} = l_R(\omega R_i/\omega R_{i+1}) = e-1$. Let now $i = n-1$. By definition $r_n = \#[\nu(\omega R_{n-1}) \setminus \nu(\gamma)]$. Since $\nu(\omega R_{n-1}) = \nu(\omega) + (n-1)e = \langle (n-1)e, (n-1)e+1, ..., ne-a-2, ne-a, ..., ne-1\rangle$, we see that only the first $e-a-1$ elements are smaller than $c = ne - a$ and we conclude that $r_n = e-a-1$.

iii) For $a = 0, ..., e-3$ we see that $1 \in \nu(\omega)$, then $\omega^2 = \overline{R}$. For $a = e-2$, by item $i)\omega = \langle 0, 2, 3, ..., e-1\rangle$, then $\omega^2 = \{0, 2, \rightarrow\}$.

iv) For $a = 0, ..., e-3$ and for $i = 0, ..., n-2$, using $iii)$ we get $t_{i+1} = l_R(R_i\overline{R}/R_{i+1}\overline{R}) = e$. For $a = e-2$ and for $i = 0, ..., n-2$, we have $\nu(\omega^2 R_i) \setminus \nu(\omega^2 R_{i+1}) = \{0, 2, ..., e-1, e+1\} + ie$ and we get again $t_{i+1} = e$. It remains to compute the last component $t_n = \#[\nu(\omega^2 R_{n-1}) \setminus \nu(\gamma)]$. For $a = 0, ..., e-3$, $\nu(\omega^2 R_{n-1}) = \nu(R_{n-1}\overline{R}) = \{(n-1)e, \rightarrow\}$; in this set the elements $< c$ are $e-a$, so $t_n = e-a$. For $a = e-2$, we have by $i)r_n = 1$, then by Prop. 3.2 also $t_n = 1$.

v) The thesis follows from $iii)$, by applying Lemma 2.3. □

Proposition 5.4. *Let* $e \geq 3$.

i) For $0 \leq a < e-2$,
$t.s.(R) = [e-1, ..., e-1, e-1-a] \Longleftrightarrow t.s.(\theta_D) = [e, e, ..., e, e-a]$.
ii) $t.s.(R) = [e-1, ..., e-1,\ 1] \Longleftrightarrow t.s.(\theta_D) = [e, e, ..., e,\ 1]$.

Proof. Both implications $\Longrightarrow$ follow from Prop. 5.2 and Lemma 5.3.

i) $\Longleftarrow$ Suppose $0 \leq a < e-2$ and $t.s.(\theta_D) = [e, e, ..., e, e-a]$. By Prop. 4.4 $r_n = \delta - \sum_{i=1}^{n-1} r_i = e - a - 1$ and by hypothesis $\delta + l_R(R/\theta_D) = ne - a$. Then $ne - a - l_R(R/\theta_D) - \sum_{i=1}^{n-1} r_i < e - a \Longrightarrow \sum_{i=1}^{n-1} r_i > (n-1)e - l_R(R/\theta_D) = (n-1)(e-1) + (n - l_R(R/\theta_D)) - 1$, i.e. $\sum_{i=1}^{n-1} r_i \geq (n-1)(e-1) + (n - l_R(R/\theta_D))$. On the other hand $\sum_{i=1}^{n-1} r_i \leq (n-1)r_1 \leq (n-1)(e-1)$. The only possibility is $\sum_{i=1}^{n-1} r_i = (n-1)(e-1)$ and $l_R(R/\theta_D) = n$, i.e. $\theta_D = t^c\overline{R}$. Hence $r_i = e-1$ for $i = 1, \ldots, n-1$ and $r_n = e - a - 1$.

ii) $\Longleftarrow$ Suppose $t.s.(\theta_D) = [e, e, ..., e, 1]$. By Lemma 4.2 $r_n = 1$. As in the above item we find $\sum_{i=1}^{n-1} r_i = (n-1)(e-1) + n - l_R(R/\theta_D) - 1$. Hence $n - l_R(R/\theta_D) - 1 \leq 0$, i.e. either $n - l_R(R/\theta_D) = 0$ or $n - l_R(R/\theta_D) = 1$. In the first case $\theta_D = \gamma$, moreover $\delta = \sum_{i=1}^{n-1} r_i + 1 = (n-1)(e-1) \Longrightarrow \delta = ne - n - e + 1 = ne - c + \delta - e + 1 \Longrightarrow c - 1 = ne - e$, which is a contradiction. The other possibility leads to $l_R(\theta_D/\gamma) = 1$ and $\sum_{i=1}^{n-1} r_i = (n-1)(e-1)$, hence $r_i = e-1$ for every $i = 0, ..., n-1$. $\square$

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